Tutorial 2

Fundamentals

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Checking if Symmetric Matrix is PD or PSD by Computing its Eigenvalues

Definition Any number λ such that the equation $Ax = \lambda x$ has a non-zero vector-solution x is called an eigenvalue (or a characteristic root) of the equation.

A symmetric matrix is PD if its eigenvalues $\lambda_i > 0$ for all $i = 1, 2, \ldots, n$ and PSD if $\lambda_i \geq 0$.

How to calculate eigenvalues: $Ax - \lambda x = 0 \Rightarrow (A - \lambda I)x = 0$. Since x is non-zero, the determinant of $(A - \lambda I)$ should vanish. Therefore all eigenvalues can be calculated as roots of the equation (which is often called the characteristic equation of A):

$$\det(A - \lambda I) = 0.$$

Example

Consider the Hessian matrix

$$\nabla^2 f(x) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Computing eigenvalues

$$\det(\nabla^2 f(x) - \lambda I) = \begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{pmatrix} = (5-\lambda)(\lambda^2 - 6\lambda + 8) = (5-\lambda)(\lambda-2)(\lambda-4) = 0.$$

Therefore, the eigenvalues are $\lambda = 2$, $\lambda = 4$ and $\lambda = 5$. As ll of them are strictly positive, the Hessian is positive definite (PD).

Properties of Convex Functions

- if f is convex function, its sublevel set $f(x) \leq \alpha$ is convex;

- positive multiple of convex function is convex:

  f convex, $\alpha \geq 0 \implies \alpha f$ convex

- sum of convex functions is convex:

  f_1, f_2 convex $\implies f_1 + f_2$ convex

- pointwise maximum of convex functions is convex:

  f_1, f_2 convex $\implies \max\{f_1(x), f_2(x)\}$ convex

  (corresponds to intersections of epigraphs)

- affine transformation of domain:

  f convex $\implies f(Ax + b)$ convex

Composition Rules

Composite function

$$f(x) = h(g(x))$$

is convex if:

- g convex; h convex nondecreasing

- g concave; h convex nonincreasing

Proof (differentiable functions, $x \in \Re$):

$$f'' = h''(g')^2 + g''h'$$

Examples:

- $f(x) = e^{g(x)}$ is convex if g is convex

- $f(x) = 1/g(x)$ is convex if g is concave, positive

- $f(x) = g(x)^p,\ p \geq 1$ is convex if $g(x)$ is convex, positive